MATH 2010 B Advanced Calculus I. Lecture Notes Week 6

Martin Li

Last time ... 2D limits, polar coordinates.

Q: How to determine if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and how to calculate the limit if it exists?

To show that a 2D-limit does not exist

- find a path approaching (a,b) s.t. the 1D-limit of f restricted to the path does not exist.
- find two different paths approaching (a,b) s.t. the 1D-limits of f restricted to these paths are different.

How to calculate limits

- Substitution: E.g. $\lim_{(x,y)\to(0,0)} \frac{2x+y+2}{y^2+1} = \frac{0+0+2}{0+1} = 2$.
- Simplification: E.g. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} (x+y) = 0$.
- Sandwich Theorem (Squeezing)

  If $g(x,y) \leq f(x,y) \leq h(x,y)$ for all (x,y) near (a,b),

  and $\lim_{(x,y)\to(a,b)} g(x,y) = L = \lim_{(x,y)\to(a,b)} h(x,y)$.

  then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

  Special case: If $|f(x,y)| \leq g(x,y)$ for all (x,y)

  and $\lim_{(x,y)\to(0,0)} g(x,y) = 0$

  then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Example: Find $\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right)$.

Sol: Since $|\cos\theta| \le 1$, we have

$$\left| x\cos\left(\frac{1}{x^2+y^2}\right)\right| \le |x| \qquad \forall (x,y)\in\mathbb{R}^2.$$

and $\lim_{(x,y)\to(0,0)} |x| = 0$, by Sandwich Theorem,

$$\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right) = 0.$$

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· Using polar coordinates:

Fact: $\lim_{(x,y)\to(0,0)} f(x,y) = \underbrace{\lim_{r\to0} f(r,\theta)}_{\text{1D-limit}}$.

Example: Find $\lim_{(x,y)\to(0,0)} \frac{x^3+xy^3}{x^2+y^2}$.

Sol: Use polar coordinates, let $x = r\cos\theta$, $y = r\sin\theta$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3+xy^3}{x^2+y^2} = \lim_{r\to0} \frac{r^3\cos^3\theta + (r\cos\theta)(r^3\sin^3\theta)}{r^2}$$

$$= \lim_{r\to0} r\cos^3\theta + r^2\cos\theta\sin^3\theta$$

$$= 0 \quad \text{(independent of } \theta!\text{)}$$

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Example: $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$

$$= \cos^2\theta - \sin^2\theta \quad \text{(depends on } \theta\text{)}$$

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2D-limit does not exist!

A more complicated example

Let

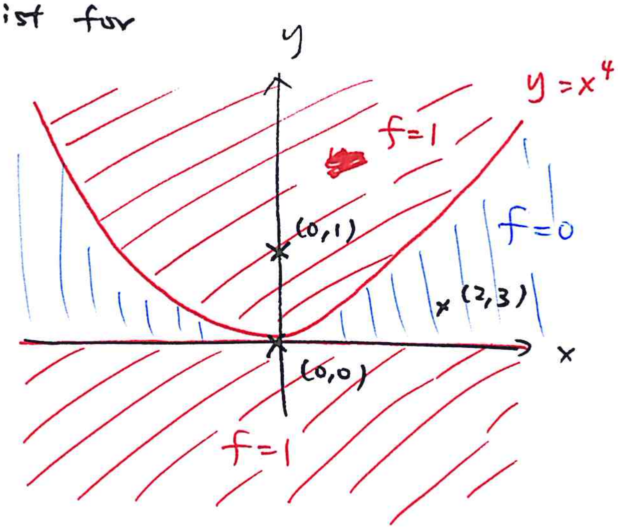
$$f(x,y) = \begin{cases} 1 & , y \geq x^4 \\ 1 & , y \leq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Find or show that the limit does not exist for

(a) $\lim_{(x,y)\to(0,1)} f(x,y)$

(b) $\lim_{(x,y)\to(2,3)} f(x,y)$

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$



Sol: (a) limit exists and $=1$ since $f(x,y) \equiv 1$ near $(0,1)$

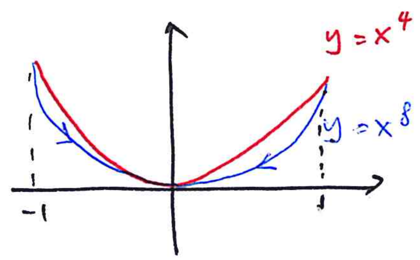
(b) limit exists and $=0$ since $f(x,y) \equiv 0$ near $(2,3)$

(c) limit does not exists since

$$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} f(x,y) = 1$$

$$\neq$$

but $$\lim_{\substack{(x,y)\to(0,0) \\ y=x^8}} f(x,y) = 0$$



Note: In this example,

$$\lim_{\substack{(x,y)\to(0,0) \\ y=kx}} f(x,y) = 1 \quad \Rightarrow$$ Consider lines of different slopes is NOT enough!

Partial Derivatives

Q: How to "differentiate" a multi-variable function, say

$$f(x,y) = x^2 + 2y \ ?$$

Idea: By "freezing" all variables except one, we can treat it as a 1-variable function, hence apply usual single variable calculus:

Freeze $y = y_0$, $\quad x \longmapsto f(x, y_0)$ $\quad$ 1-variable in x

$$\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) := \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

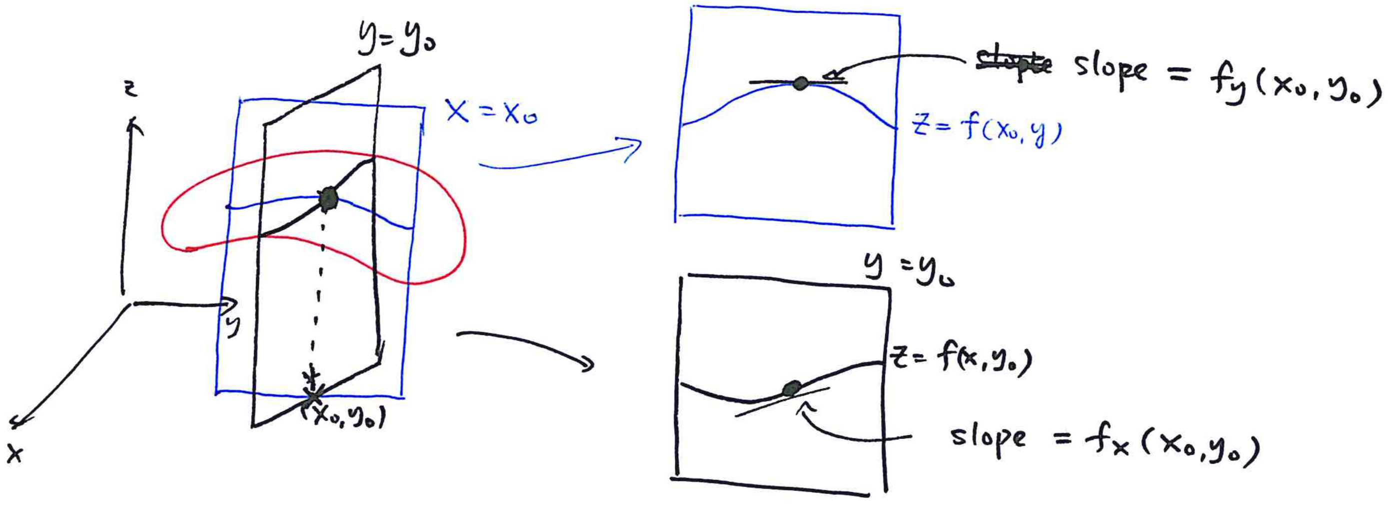
"partial derivative with respect to x"

Freeze $x = x_0$, $\quad y \longmapsto f(x_0, y)$ $\quad$ 1-variable in y

$$\frac{\partial f}{\partial y}(x_0,y_0) = f_y(x_0,y_0) := \lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0,y_0)}{k}$$

"partial derivative with respect to y"

Geometric meaning - "slicing"



How to calculate partial derivatives

- When f is defined by a single formula near (x_0, y_0)

then $\frac{\partial f}{\partial x}$ is just differentiating in x,

treating the variable y as a constant.

E.g. $f(x,y) = x^2 + 2y^2 + xy$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(2y^2) + \frac{\partial}{\partial x}(xy)$$

$$= 2x + 0 + y\frac{\partial}{\partial x}(x) = 2x + y \quad *$$

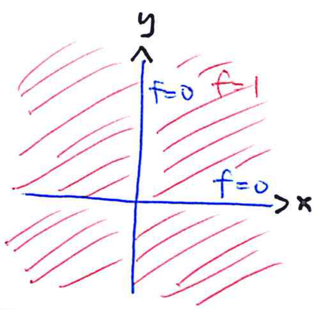
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial y}(xy)$$

$$= 0 + 4y + x\frac{\partial}{\partial y}(y) = 4y + x \quad * .$$

Fact: Knowing f_x, f_y does not give us too much information about f sometimes, e.g.

$$f_x, f_y \text{ exists at } (0,0) \not\Rightarrow f \text{ continuous at } (0,0)$$

Example: Consider



$$f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0. \end{cases}$$

then,

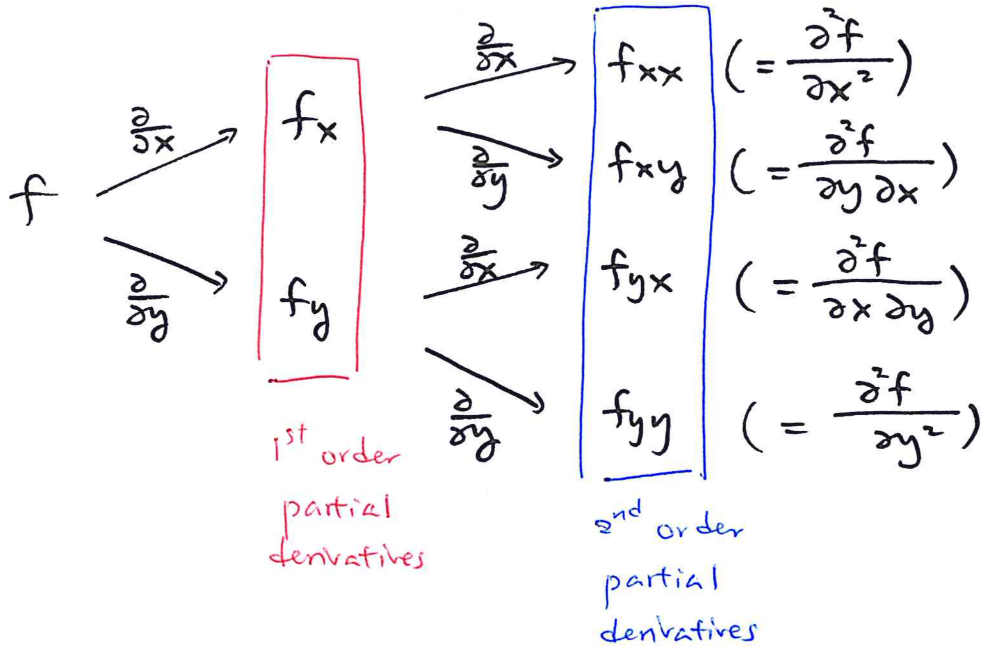
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

but f is not continuous at $(0,0)$.

e.g. along $y = x$, $f \to 1$ as $(x,y) \to (0,0)$ but $f(0,0) = 0 \neq 1$.

Higher Derivatives



Note: The "order" appears differently in these two notations.

E.g. 1 $f(x,y) = 2x^2 + 3xy + 4y^2$

1st order:

$$\begin{cases} f_x = 4x + 3y \\ f_y = 3x + 8y \end{cases}$$

2nd order:

$$\begin{cases} f_{xx} = (f_x)_x = 4 \\ f_{xy} = (f_x)_y = 3 \\ f_{yx} = (f_y)_x = 3 \\ f_{yy} = (f_y)_y = 8 \end{cases}$$

(f_{xy} and f_{yx}:) SAME! "coincidence?"

E.g. 2: $u(x,y) = \ln\sqrt{x^2+y^2}$, $(x,y) \neq (0,0)$

Check that u satisfies the Laplace equation:

$$\boxed{\Delta u := u_{xx} + u_{yy} = 0}$$

Physical meaning: This represents equilibrium heat distribution if $u(x,y)$ = temperature at (x,y).

Sol: $u(x,y) = \ln (x^2+y^2)^{\frac{1}{2}} = \frac{1}{2} \ln(x^2+y^2)$

$$\begin{cases} u_x = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)(2x) = \frac{x}{x^2+y^2} \\ u_y = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)(2y) = \frac{y}{x^2+y^2} \end{cases}$$

$$\left.\begin{aligned} u_{xx} &= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} \\ u_{yy} &= \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \end{aligned}\right\} \Rightarrow u_{xx} + u_{yy} = 0 .$$

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Side remark:

$$\left.\begin{aligned} u_{xy} &= (u_x)_y = \frac{-2xy}{(x^2+y^2)^2} \\ u_{yx} &= (u_y)_x = \frac{-2xy}{(x^2+y^2)^2} \end{aligned}\right\rangle$$ same again! So many "coincidence"?

Mixed Derivative Theorem

If f is in C^2 (ie. all the 2^{nd} order partial derivatives exist and are continuous)
↳ as 2D-functions

then

$$\boxed{f_{xy} = f_{yx}}$$

ie. we can switch the order of derivatives